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Abstract

- In many applications, it is desirable to estimate a state (e.g. strain, acceleration, temperature) that cannot be directly measured. In such applications, what's known as an "observer" is designed to assimilate measured data with a dynamic system model to produce an estimate of the immeasurable system states. A specific example would be estimating the strain at some position in a wing, when only acceleration measurements at the fuselage are available. For this tutorial, a control systems oriented overview of observers in general and Kalman filters specifically will be discussed. Theoretical foundations will be covered, and several examples will be worked. Extensions to nonlinear systems (a.k.a. the extended Kalman filter) will also be discussed, with hardware demonstrations, if time permits. Attendees are encouraged to bring their own examples of applications that require or could benefit from state estimation.

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Kalman Filtering

A Brief Introduction

Matt Bement, X-3-LGRN

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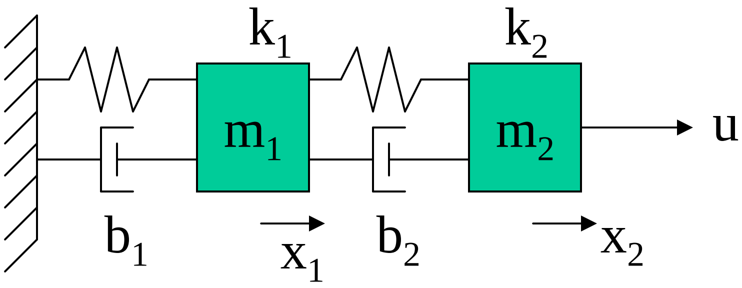
Outline

- Motivation
- Basics of Dynamic Systems
- Controllers and Observers
- Kalman filtering
  - Examples
- Implementation issues
  - Examples
- Ensemble Kalman filter
- ~~Hardware demonstration~~ (Tuesday 9:00)
- Derivation

Motivation



Example 2 DOF system:

Given x_1, estimate x_2

Given $\ddot{x}_1$, estimate x_1

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Representation of dynamic systems

- Nonlinear

$$u \longrightarrow \boxed{\begin{aligned}\dot{x} &= f(x,u,t)\\ y &= g(x,u,t)\end{aligned}} \longrightarrow y$$

- Linear

$$u \longrightarrow \boxed{\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}} \longrightarrow y$$

$$u \longrightarrow \boxed{P} \longrightarrow y$$

Controllability and Observability

Intuitive definition:

- Controllability: The degree to which the input, u, can influence each state, x
- Observability: The degree to which each state, x, can influence the output, y

Semi-intuitive definition (for single input, single output systems:

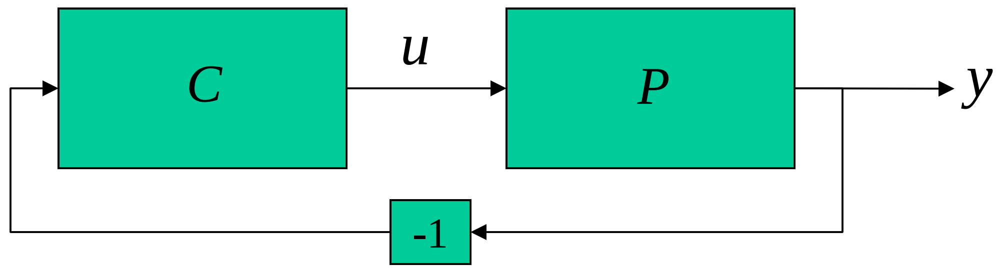
- Uncontrollable: B is orthogonal to one of the eigenvectors of A
- Unobservable: C is orthogonal to one of the eigenvectors of A

Mathematical definition:

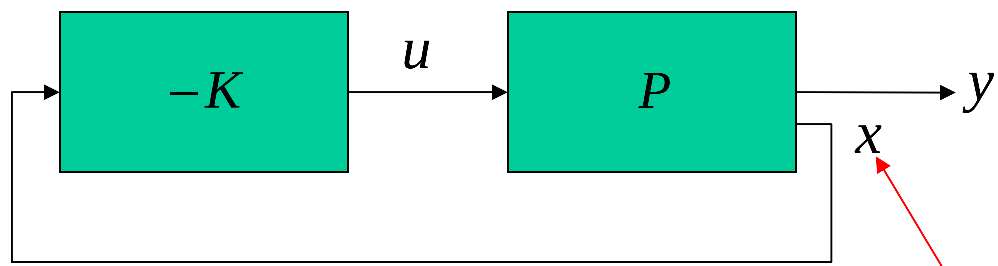
- Uncontrollable: $rank\left(\left[B \;\; AB \;\; A^2B \ldots A^{n-1}B\right]\right) < n$ Unobservable: $rank\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}\right) < n$

Feedback control

Output feedback:



State feedback:

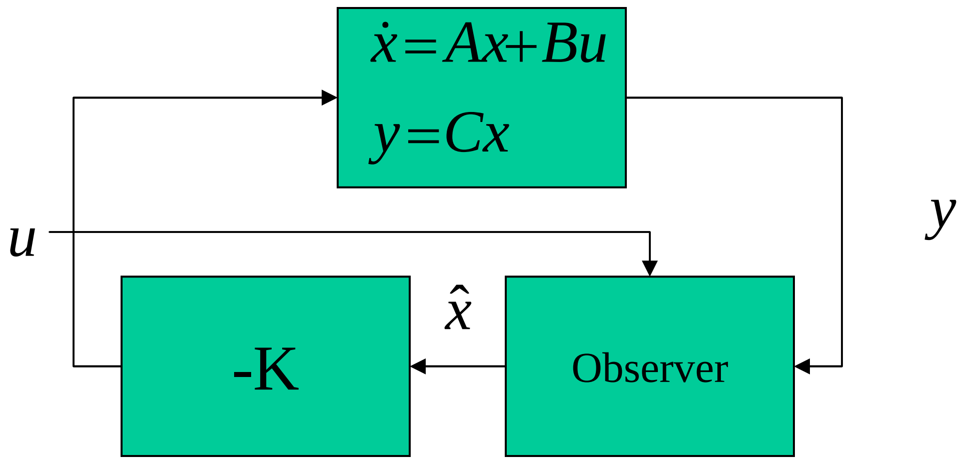


$$\dot{x} = (A - BK)x$$

Where does this come from?

If A,B is controllable, then we can make the closed loop system anything we want!

Observers (estimators)



$$\dot{\hat{x}} = A\hat{x} + Bu + g\left(y - C\hat{x}\right)$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(y - C\hat{x}\right)$$

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(Cx - C\hat{x}\right)$$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = \left(A - LC\right)\tilde{x}$$

If A,C is observable, then we can make the estimation error decay as fast as we want!

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Tradeoffs in control and estimation

For control, there's a tradeoff between x (performance) and u (control effort)

What tradeoff, if any, is there in estimation?

$$\dot{x} = Ax + Bu + w$$

$$y = Cx + v$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\left(Cx + v - C\hat{x}\right)$$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = \left(A - LC\right)\tilde{x} + w - Lv$$

Observers

- Kalman filter (and associated variants)
- H_∞
- Sliding mode
- Neural network
- …

Kalman Filtering

- Named after Rudolf Kalman
- Resources
  - Kalman, R.E. (1960) *“A new approach to linear filtering and prediction problems”*, Journal of Basic Engineering, vol 82, no. 1, pp. 35-45
  - Gelb, A (1974). *Applied Optimal Estimation*, MIT Press
  - http://www.cs.unc.edu/~welch/kalman/
    - UCLA Extension short course on Kalman Filtering
    - Optimal State Estimation short course by Dan Simon (next course is in August)

Kalman filter algorithm

Goal: Minimize the variance of the estimation error

$$\min\left(Trace\left(\mathrm{cov}\left(\tilde{x}\right)\right)\right)$$

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1} + w_{k-1}$$
$$y_k = Cx_k + v_k$$

Predict

$$1.\ \hat{x}^-{}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}$$
$$2.\ P^-{}_k = A_{k-1}P_{k-1}A_{k-1}{}^T + Q_{k-1}$$

Update

$$3.\ L_k = P^-{}_k C_k{}^T\left[C_k P^-{}_k C_k{}^T + R_k\right]^{-1}$$
$$4.\ \hat{x}_k = \hat{x}^-{}_k + L_k\left(y_k - C_k\hat{x}^-{}_k\right)$$
$$5.\ P_k = \left(I - L_k C_k\right)P^-{}_k$$

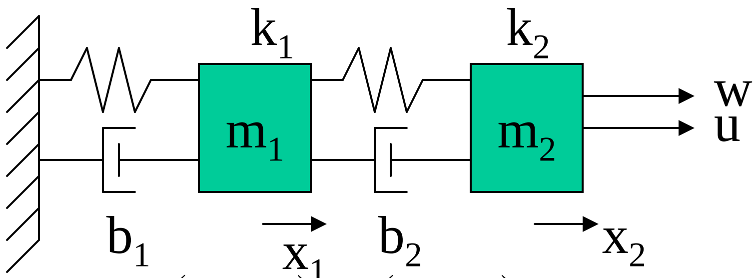
$$Q_k = E(w_k w_k{}^T)$$
$$R_k = E(v_k v_k{}^T)$$

Example 1: 2 DOF System

m1=m2=1

k1=k2=5

b1=b2=1



var(w)=0.01

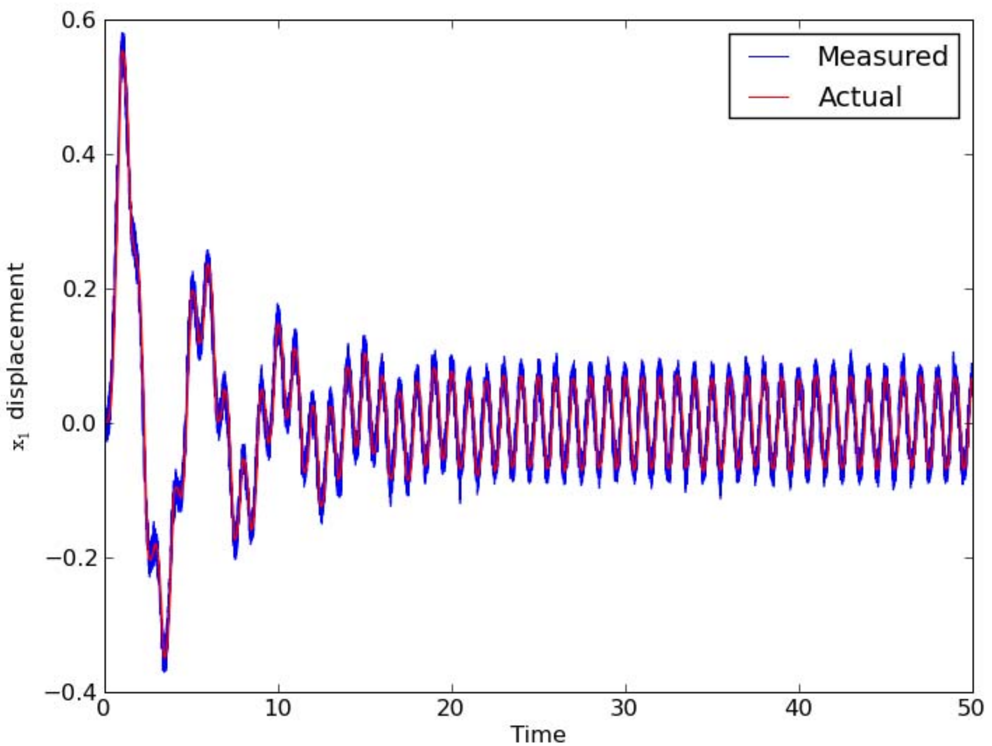
var(v)=0.0001

u=10*sin(2*pi*4*t)

$$m_1\ddot{x}_1 = -(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2$$

$$m_2\ddot{x}_2 = k_2x_1 + b_2\dot{x}_1 - k_2x_2 - b_2\dot{x}_2 + u + w$$

$$y = x_1 + v$$



Example 1: 2 DOF System

$$\begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{b_1+b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} \\ 0 & 0 & 1 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + v$$

$$1.\ \hat{x}^-{}_k = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}$$

$$2.\ P^-{}_k = A_{k-1}P_{k-1}A_{k-1}{}^T + Q_{k-1}$$

$$3.\ L_k = P^-{}_k C_k{}^T \left[C_k P^-{}_k C_k{}^T + R_k \right]^{-1}$$

$$4.\ \hat{x}_k = \hat{x}^-{}_k + L_k \left(y_k - C_k \hat{x}^-{}_k \right)$$

$$5.\ P_k = \left(I - L_k C_k \right) P^-{}_k$$

$$\dot{x} = Ax + Bu + Gw$$

$$y = Cx + v$$

$$x_k = (I + \Delta t \cdot A)x_{k-1} + (\Delta t \cdot B)u_{k-1} + (\Delta t \cdot G)w_{k-1}$$

$$y_k = Cx_k + v_k$$

Example 1: 2 DOF System

$$1.\ \hat{x}^{-}_{k} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}$$

$$2.\ P^{-}_{k} = A_{k-1}P_{k-1}A_{k-1}^{T} + Q_{k-1}$$

$$3.\ L_k = P^{-}_{k}C_k^{T}\left[C_k P^{-}_{k} C_k^{T} + R_k\right]^{-1}$$

$$4.\ \hat{x}_k = \hat{x}^{-}_{k} + L_k\left(y_k - C_k\hat{x}^{-}_{k}\right)$$

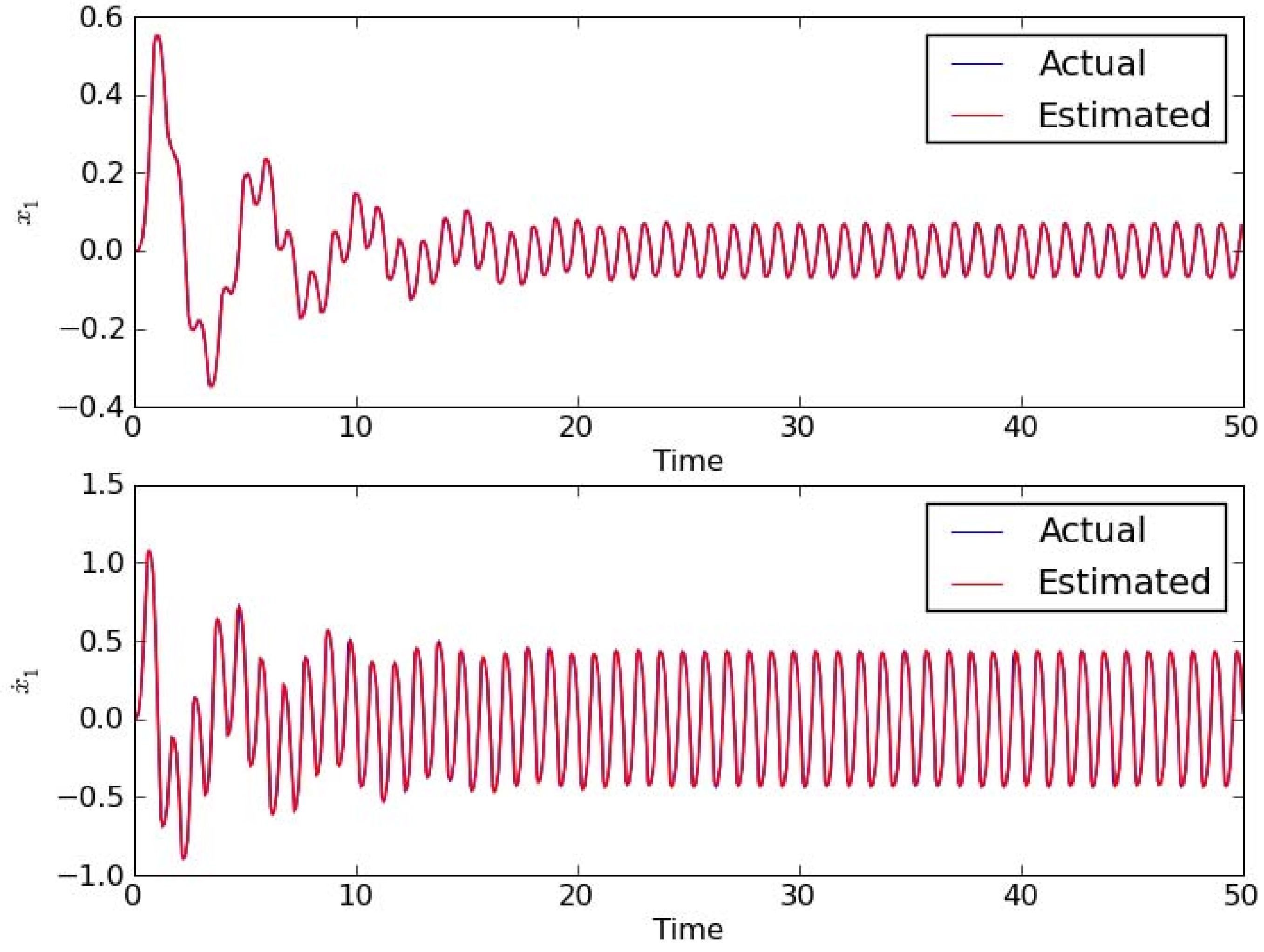
$$5.\ P_k = \left(I - L_k C_k\right)P^{-}_{k}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{cov}\left(w \cdot \Delta t / m_2\right) = 6.326e-6 \end{bmatrix}$$

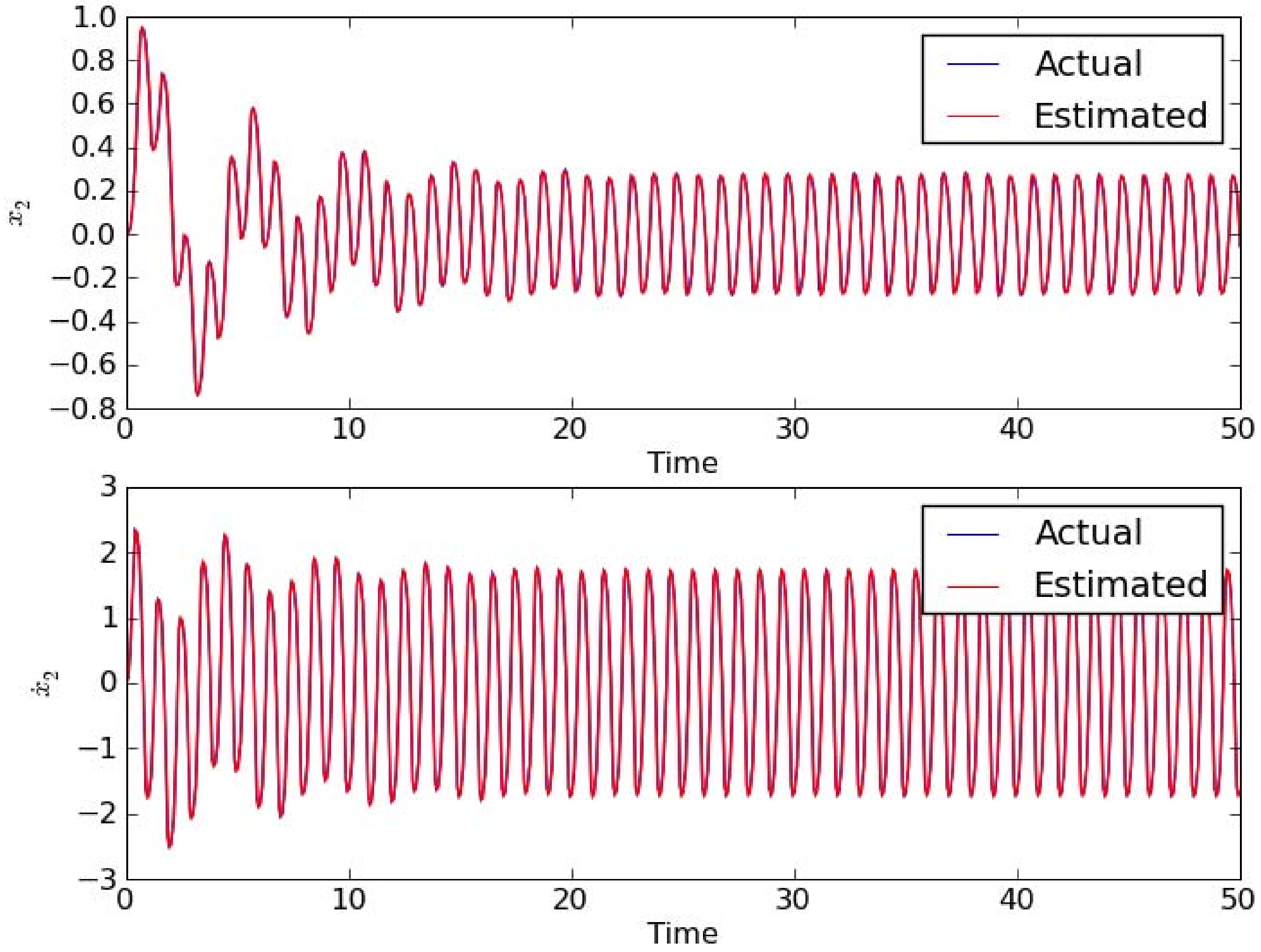
$$R = \mathrm{cov}(v) = 1.0e-4$$

$$P_0 = E\left(\tilde{x}_0 \tilde{x}_0^{T}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1: 2 DOF System



Example 1: 2 DOF System



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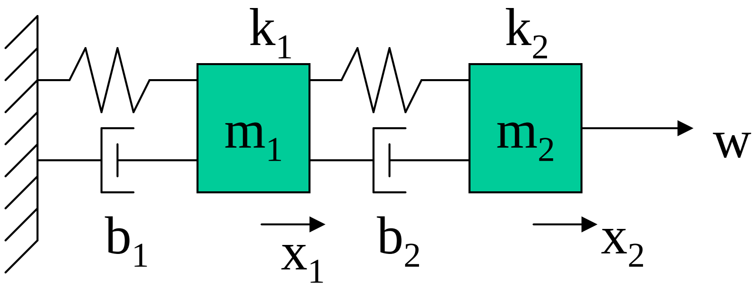
Example 2 – Displacement estimation from acceleration

```
m1=m2=1
k1=k2=5
b1=b2=1
```

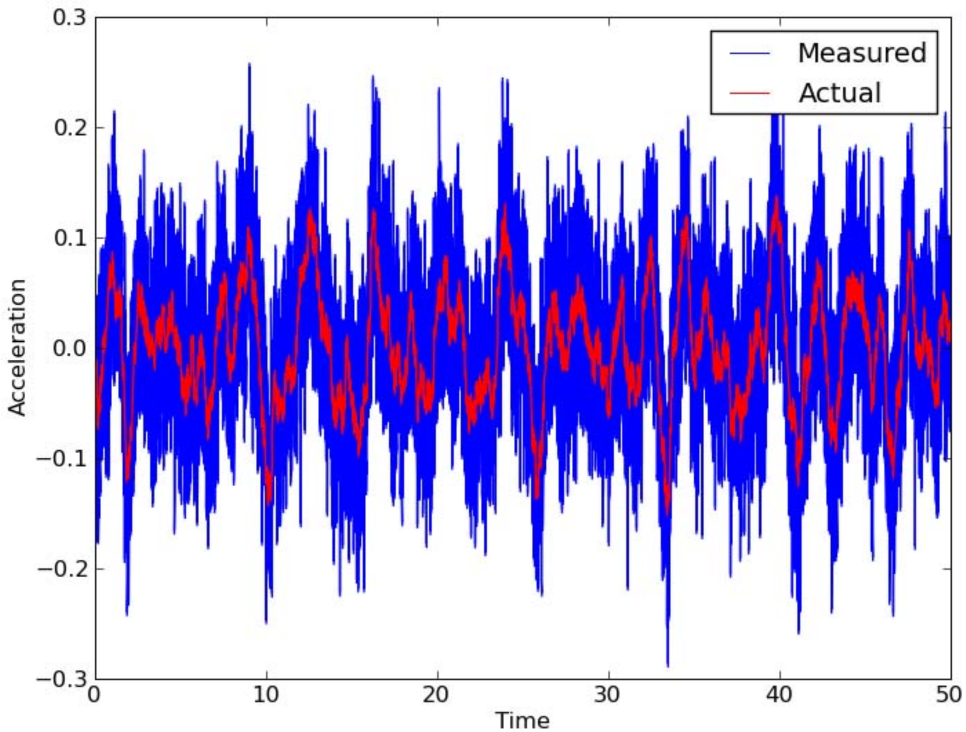



```
var(w)=1
var(v)=0.0025
```

$$m_1\ddot{x}_1 = -(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2$$

$$m_2\ddot{x}_2 = k_2x_1 + b_2\dot{x}_1 - k_2x_2 - b_2\dot{x}_2 + w$$

$$y = \ddot{x}_1 = \frac{-(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2}{m_1} + v$$



Example 2 – Displacement estimation from acceleration

$$1.\ \hat{x}^{-}{}_{k} = A_{k-1}\hat{x}_{k-1} + B_{k-1}u_{k-1}$$

$$2.\ P^{-}{}_{k} = A_{k-1}P_{k-1}A_{k-1}{}^{T} + Q_{k-1}$$

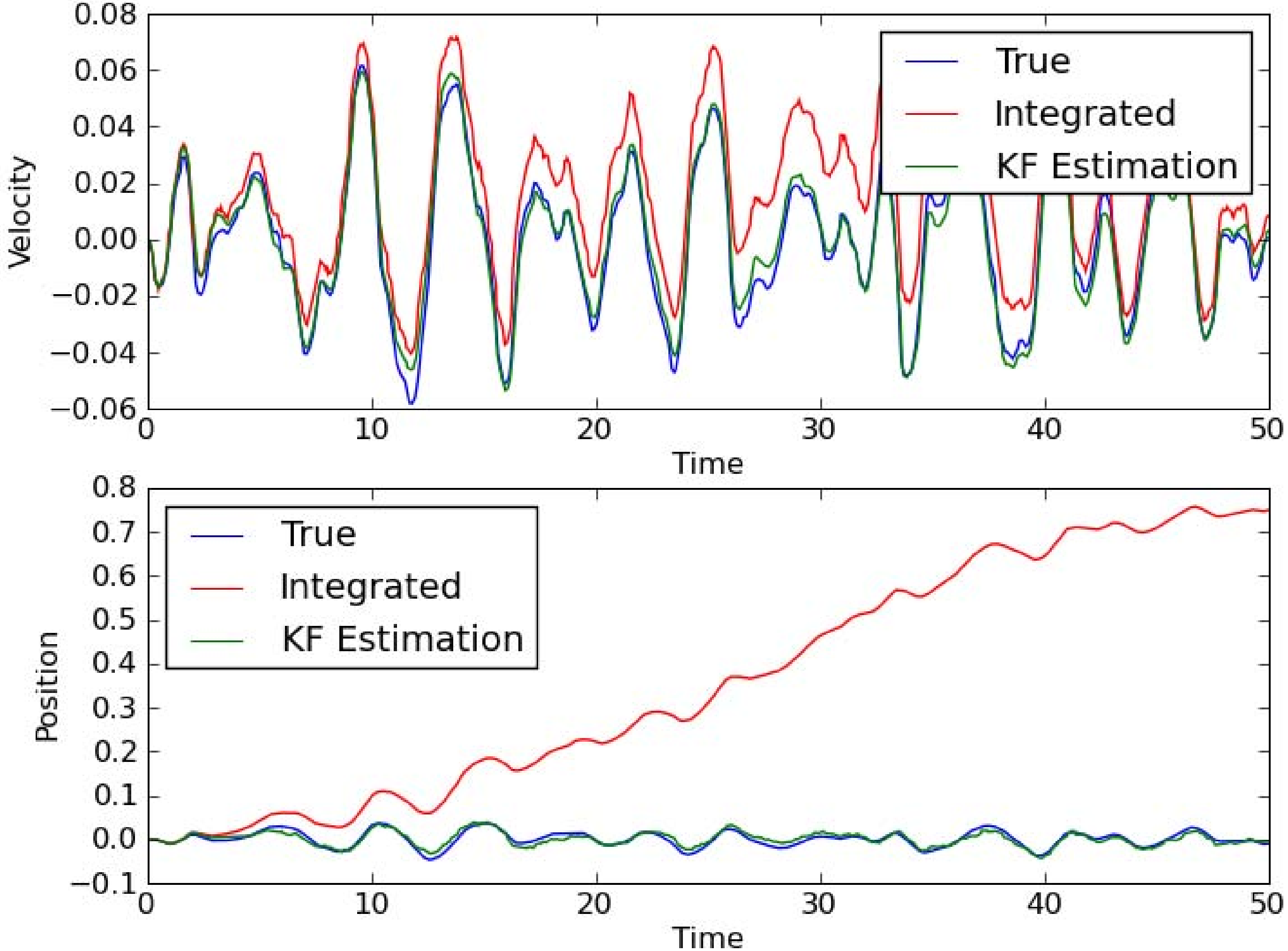
$$3.\ L_k = P^{-}{}_{k}C_k{}^{T}\left[C_k P^{-}{}_{k} C_k{}^{T} + R_k\right]^{-1}$$

$$4.\ \hat{x}_k = \hat{x}^{-}{}_{k} + L_k\left(y_k - C_k\hat{x}^{-}{}_{k}\right)$$

$$5.\ P_k = \left(I - L_k C_k\right)P^{-}{}_{k}$$

$$C = \begin{bmatrix} -\dfrac{k_1+k_2}{m_1} & -\dfrac{b_1+b_2}{m_1} & \dfrac{k_2}{m_1} & \dfrac{b_2}{m_1} \end{bmatrix}$$

Example 2 – Displacement estimation from acceleration



Extended Kalman filter

$$x_k = f_{k-1}\left(x_{k-1}, u_{k-1}\right) + w_{k-1}$$
$$y_k = h_k\left(x_k\right) + v_k$$

$$A_k = \left.\frac{\partial f_k}{\partial x}\right|_{\hat{x}_k, u_k}$$

$$C_k = \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}^-{}_k}$$

1. $\hat{x}^-{}_k = f_{k-1}\left(x_{k-1}, u_{k-1}\right)$
2. $P^-{}_k = A_{k-1} P_{k-1} A_{k-1}{}^T + Q_{k-1}$
3. $L_k = P^-{}_k C_k{}^T \left[C_k P^-{}_k C_k{}^T + R_k\right]^{-1}$
4. $\hat{x}_k = \hat{x}^-{}_k + L_k\left(y_k - C_k \hat{x}^-{}_k\right)$
5. $P_k = \left(I - L_k C_k\right) P^-{}_k$

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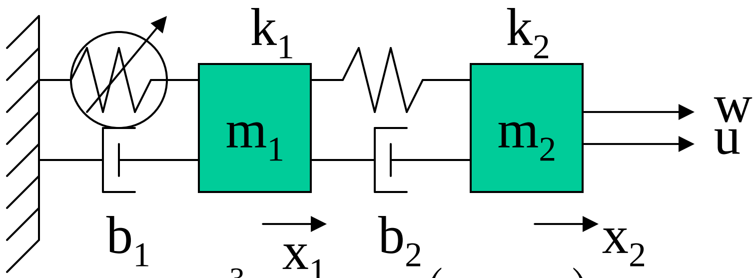
Example 3 – Estimating a parameter

```
m1=m2=1
k1=k2=5
b1=b2=1
```

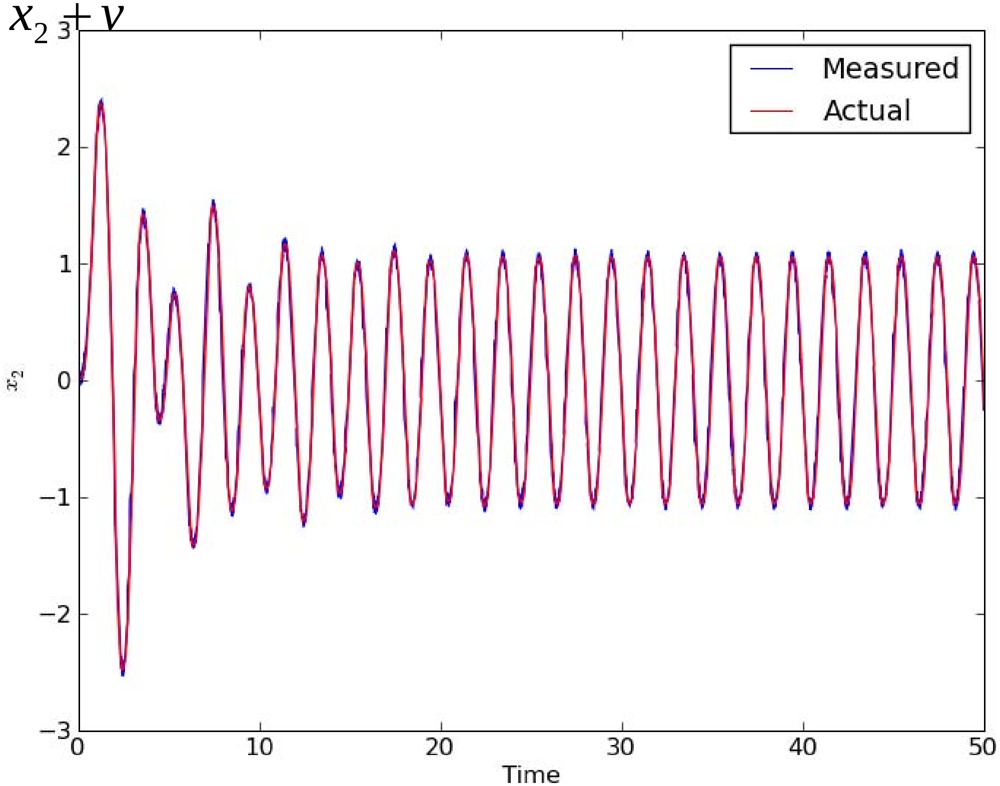



```
var(w)=0.0001
var(v)=0.0001
u=10*sin(2*pi*0.5*t)
```

$$m_1\ddot{x}_1 = -k_1x_1 - k_{nl}x_1^{\,3} - k_2x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2$$

$$m_2\ddot{x}_2 = k_2x_1 + b_2\dot{x}_1 - k_2x_2 - b_2\dot{x}_2 + u + w$$

$$y = x_2 + v$$



Example 3 – Estimating a parameter

$$m_1\ddot{x}_1 = -k_1 x_1 - k_{nl} x_1^{\,3} - k_2 x_1 - (b_1 + b_2)\dot{x}_1 + k_2 x_2 + b_2 \dot{x}_2$$
$$m_2\ddot{x}_2 = k_2 x_1 + b_2 \dot{x}_1 - k_2 x_2 - b_2 \dot{x}_2 + u + w$$
$$\dot{k}_{nl} = 0$$
$$y = x_2 + v$$

$$x = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ k_{nl} \end{bmatrix}$$

$$x_k = f_{k-1}(x_{k-1}, u_{k-1}) + w_{k-1}$$
$$y_k = h_k(x_k) + v_k$$

$$A_k = \left.\frac{\partial f_k}{\partial x}\right|_{\hat{x}_k, u_k}$$
$$C_k = \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}^-_k}$$

$$A_k = \left.\frac{\partial f_k}{\partial x}\right|_{\hat{x}_k, u_k} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-3\hat{k}_{nl,k}\hat{x}_{1,k}^{\,2} - k_1 - k_2}{m_1} & -\frac{b_1 + b_2}{m_1} & \frac{k_2}{m_1} & \frac{b_2}{m_1} & -\hat{x}_{1,k}^{\,3} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{k_2}{m_2} & \frac{b_2}{m_2} & -\frac{k_2}{m_2} & -\frac{b_2}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Delta t + I$$

$$C_k = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

1. $\hat{x}^-_k = f_{k-1}(x_{k-1}, u_{k-1})$
2. $P^-_k = A_{k-1} P_{k-1} A_{k-1}^{\,T} + Q_{k-1}$
3. $L_k = P^-_k C_k^{\,T} \left[C_k P^-_k C_k^{\,T} + R_k \right]^{-1}$
4. $\hat{x}_k = \hat{x}^-_k + L_k \left(y_k - C_k \hat{x}^-_k \right)$
5. $P_k = (I - L_k C_k) P^-_k$

Example 3 – Estimating a parameter

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{cov}\left(w \cdot \Delta t / m_2\right) = 6.3e-10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \operatorname{cov}(v) = 1.0e-4$$

$$P_0 = E\left(\tilde{x}\tilde{x}^T\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

?

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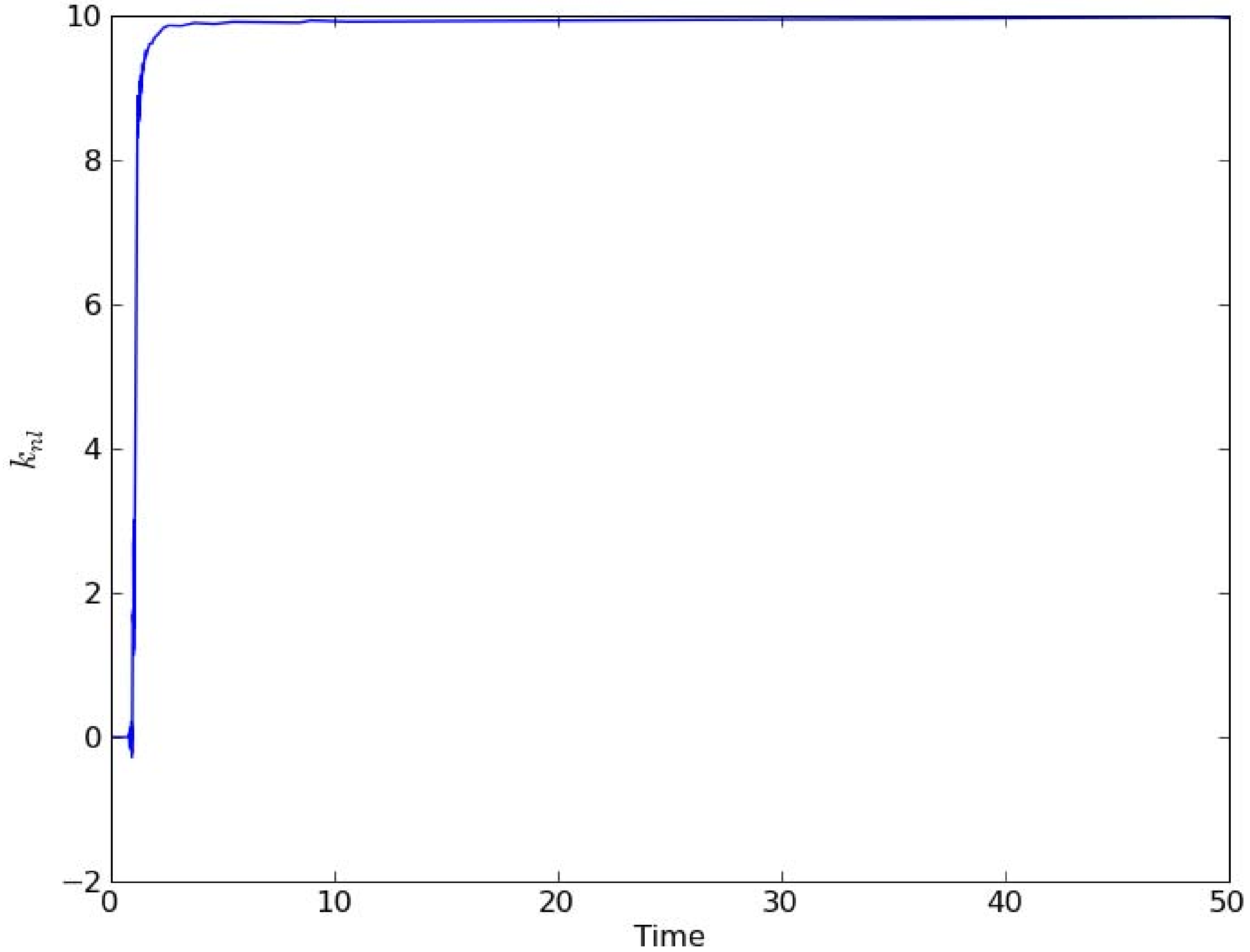
Example 3 – Estimating a parameter



Implementation issues

- Modeling errors
  - Parametric
  - Structural
  - Noise/disturbance
- Input issues
- Computational issues

Modeling errors

- What happens if (when) your model isn't correct?

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$\dot{\hat{x}} = (A + \Delta A)\hat{x} + Bu + L(Cx - C\hat{x})$$

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = (A - LC)\tilde{x} - \Delta A\hat{x}$$

- Optimality no longer guaranteed
- May not be unbiased

- Can still be very useful!

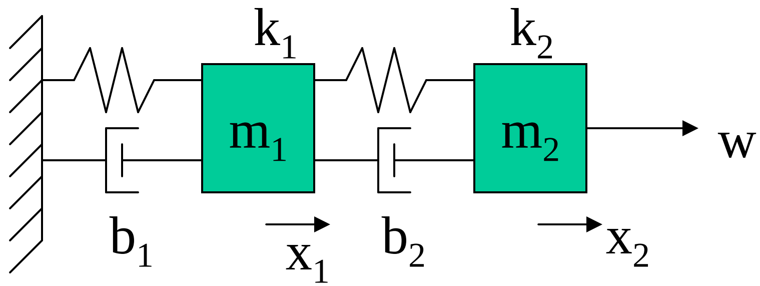
Disturbance

Sensitivity analysis

- An optimal filter may be fragile
- A suboptimal filter can be almost as good, and a lot more robust
- Perform a sensitivity analysis
  - Can indicate which parameters should be estimated with EKF
  - Can indicate how many states are needed in a model

Example 2 revisited

```
m1=m2=1
k1=8
k2=5
b1=b2=1
```

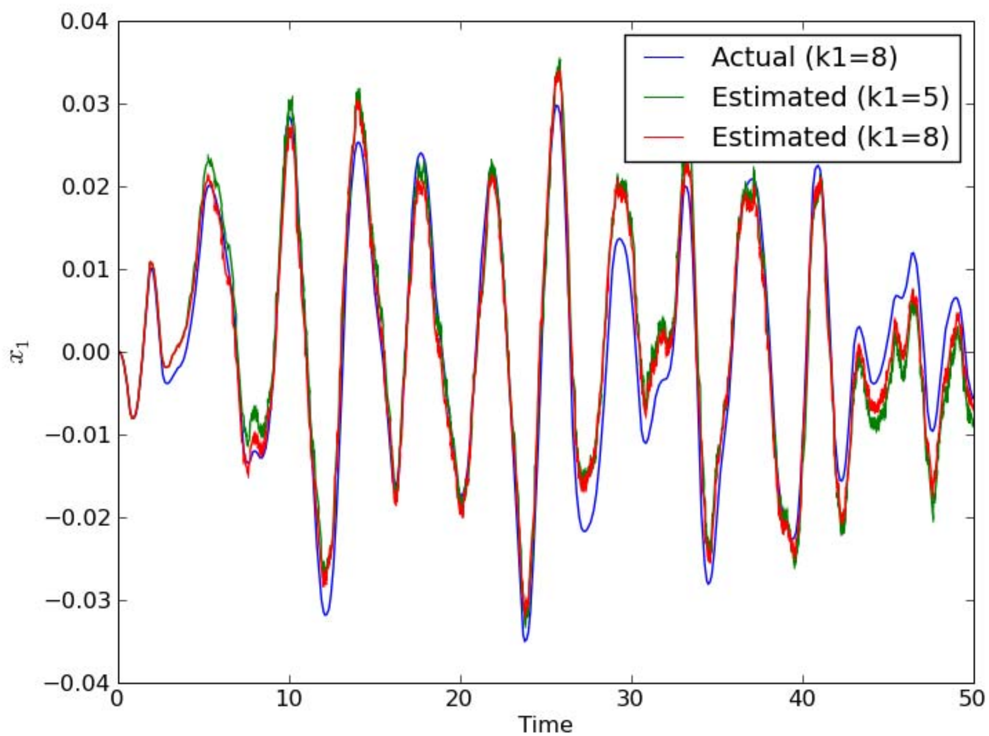



```
var(w)=1
var(v)=0.0001
```

$$m_1\ddot{x}_1 = -(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2$$

$$m_2\ddot{x}_2 = k_2x_1 + b_2\dot{x}_1 - k_2x_2 - b_2\dot{x}_2 + w$$

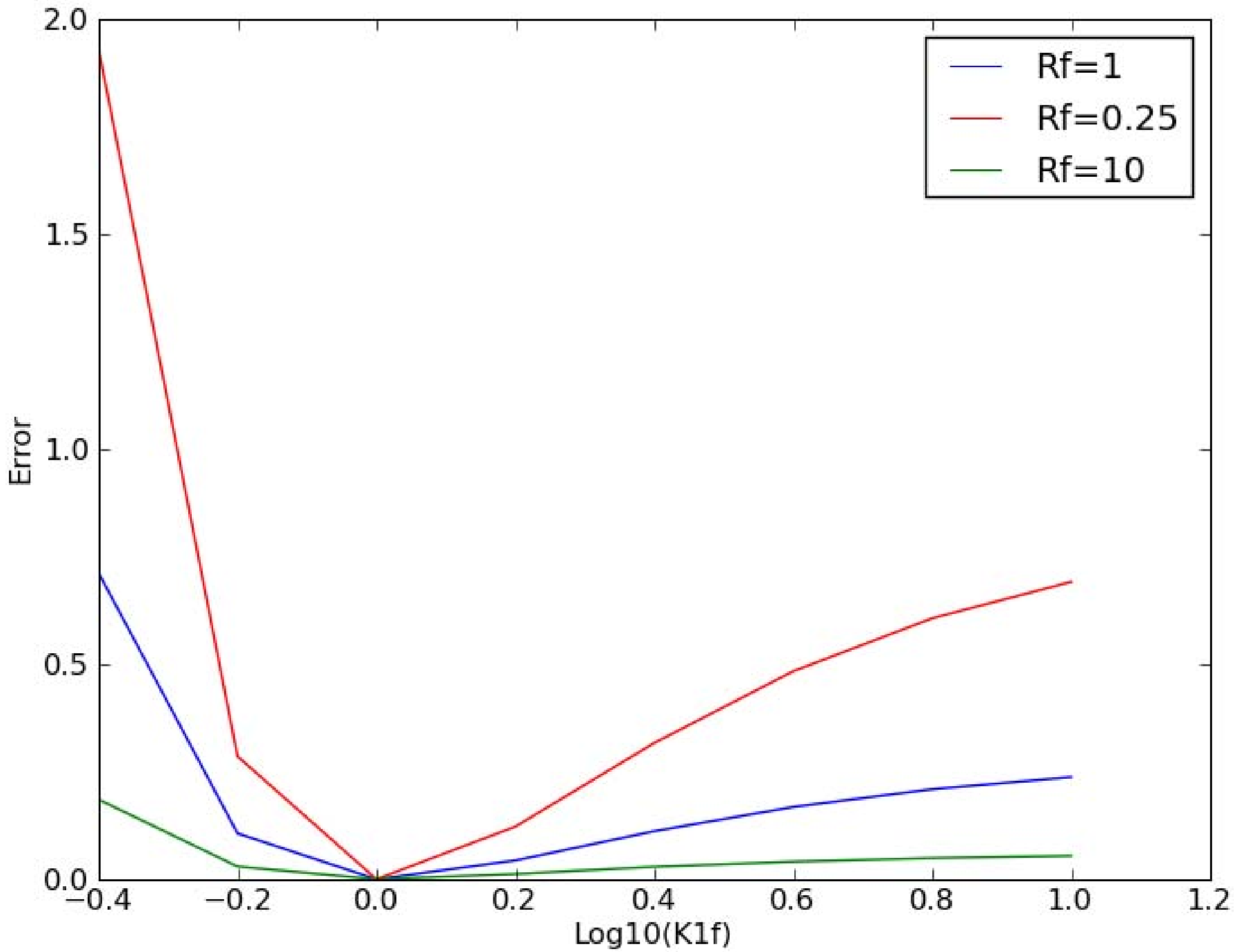
$$y = \ddot{x}_1 = \frac{-(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2}{m_1} + v$$



Error	
$\hat{k}_1=8$	1.26e-4
$\hat{k}_1=5$	3.85e-4

Example 2 revisited

- Sensitivity study:
  - k1 = [0.1 10]*5
  - R = [0.1 10]*var(v)



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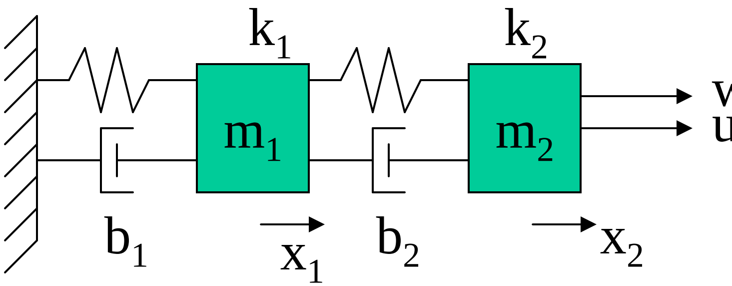
Other disturbances

m1=m2=1

k1=k2=5

b1=b2=1



var(w)=1.0

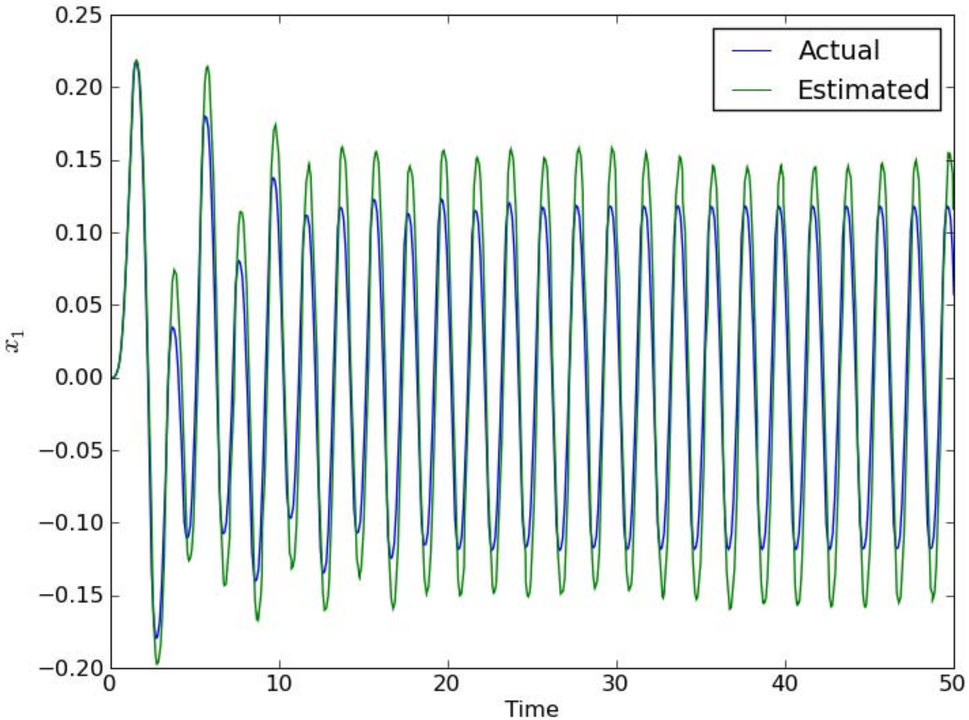
var(v)=0.0001

w=1.414*sin(2*pi*0.5*t)

$$m_1\ddot{x}_1 = -(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2$$

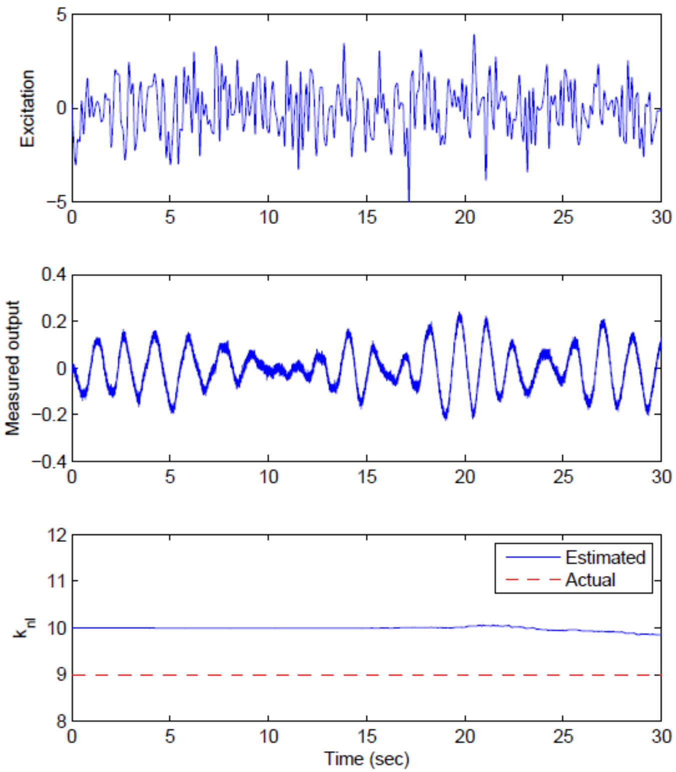
$$m_2\ddot{x}_2 = k_2x_1 + b_2\dot{x}_1 - k_2x_2 - b_2\dot{x}_2 + w$$

$$y = \ddot{x}_1 = \frac{-(k_1+k_2)x_1 - (b_1+b_2)\dot{x}_1 + k_2x_2 + b_2\dot{x}_2}{m_1} + v$$

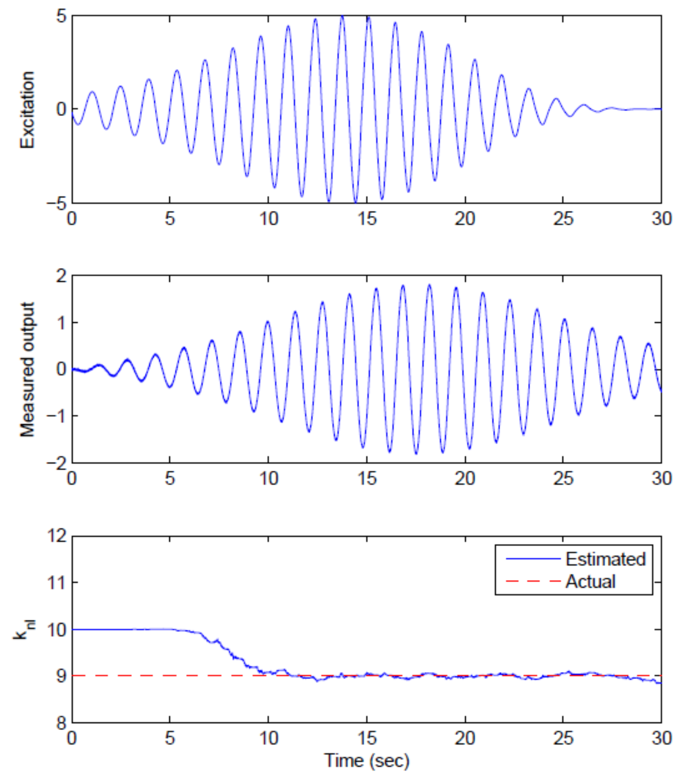


Input issues

- If you can pick the excitation, *u*, some are definitely better than others for parameter estimation
- (See Bement, Bewley in the Proceedings of the 2009 International Workshop on Structural Health Monitoring)



Naïve excitation



Optimized excitation

Computational issues

- What if $\frac{\partial f}{\partial x}$ doesn't exist or is very hard to calculate?
- What if the model has ~10^6 states (like in atmospheric modeling)?
  - P has a trillion elements!

Ensemble Kalman filter

$$0.\ \hat{x}^i{}_0 = \bar{\hat{x}}^-{}_0 + \hat{w}_0{}^i,\ i \in [1 \dots q]$$

$$1.\ \hat{x}^{i-}{}_k = f_{k-1}\left(\hat{x}^i{}_{k-1}, u_{k-1}\right) + \hat{w}^i{}_{k-1},\ E\left(\hat{w}^i{}_{k-1}\hat{w}^i{}_{k-1}{}^T\right) = Q_{k-1}$$

$$2.\ \hat{y}^{i-}{}_k = h_k\left(\hat{x}^{i-}{}_k\right) + \hat{v}^i{}_k,\ E\left(\hat{v}^i{}_{k-1}\hat{v}^i{}_{k-1}{}^T\right) = R_{k-1}$$

$$3.\ \hat{E}_{x,k} = \left[\hat{x}^{1-}{}_k - \bar{\hat{x}}^-{}_k \quad \dots \quad \hat{x}^{q-}{}_k - \bar{\hat{x}}^-{}_k\right] \leftarrow n \times q$$

$$4.\ \hat{E}_{y,k} = \left[\hat{y}^{1-}{}_k - \bar{\hat{y}}^-{}_k \quad \dots \quad \hat{y}^{q-}{}_k - \bar{\hat{y}}^-{}_k\right] \leftarrow m \times q$$

$$5.\ \hat{P}_{xy,k} = \frac{\hat{E}_{x,k}\hat{E}_{y,k}{}^T}{q-1}$$

$$6.\ \hat{P}_{yy,k} = \frac{\hat{E}_{y,k}\hat{E}_{y,k}{}^T}{q-1}$$

$$7.\ L_k = \hat{P}_{xy,k}\left[\hat{P}_{yy,k}{}^{-1}\right]$$

$$8.\ \hat{x}^i{}_k = \hat{x}^{i-}{}_k + L_k\left(y_k - \hat{y}^{i-}{}_k\right)$$

$$9.\ Estimate = \bar{\hat{x}}^i{}_k$$

$$x_k = f_{k-1}(x_{k-1}, u_{k-1}) + w_{k-1}$$
$$y_k = h_k(x_k) + v_k$$

$$A_k = \left.\frac{\partial f_k}{\partial x}\right|_{\hat{x}_k, u_k}$$
$$C_k = \left.\frac{\partial h_k}{\partial x}\right|_{\hat{x}^-{}_k}$$

$$1.\ \hat{x}^-{}_k = f_{k-1}(x_{k-1}, u_{k-1})$$

$$2.\ P^-{}_k = A_{k-1}P_{k-1}A_{k-1}{}^T + Q_{k-1}$$

$$3.\ L_k = P^-{}_kC_k{}^T\left[C_kP^-{}_kC_k{}^T + R_k\right]^{-1}$$

$$4.\ \hat{x}_k = \hat{x}^-{}_k + L_k\left(y_k - C_k\hat{x}^-{}_k\right)$$

$$5.\ P_k = (I - L_kC_k)P^-{}_k$$

Example – Lorenz system

σ=10

β=8/3

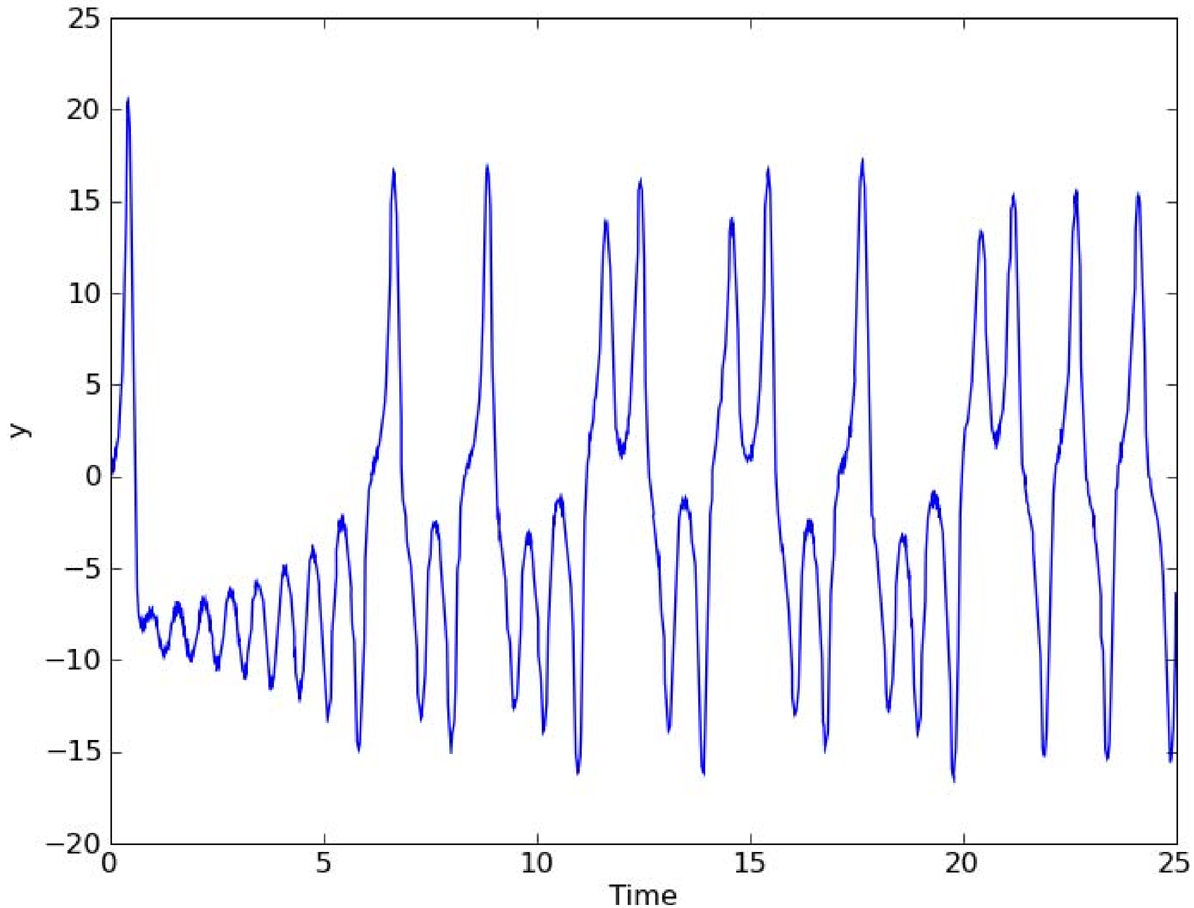
ρ=28

$$\dot{x}_1 = \sigma(x_2 - x_1) + w_1$$
$$\dot{x}_2 = x_1(\rho - x_3) - x_2 + w_2$$
$$\dot{x}_3 = x_1 x_2 - \beta x_3 + w_3$$
$$y = x_1 + v$$

cov(w)=2 I

var(v)=20

Δt=0.005



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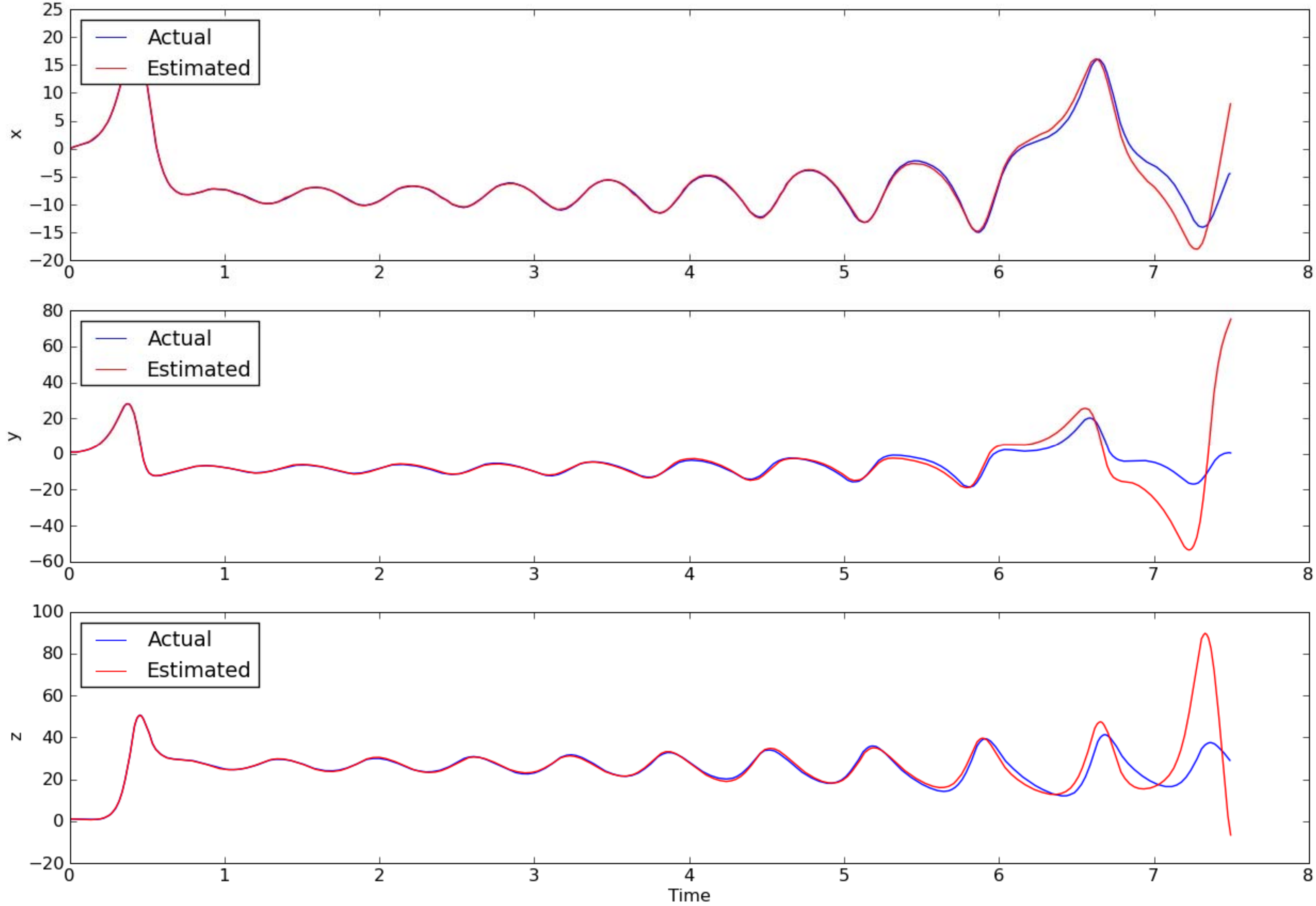
Example – Lorenz system

Extended Kalman filter diverges



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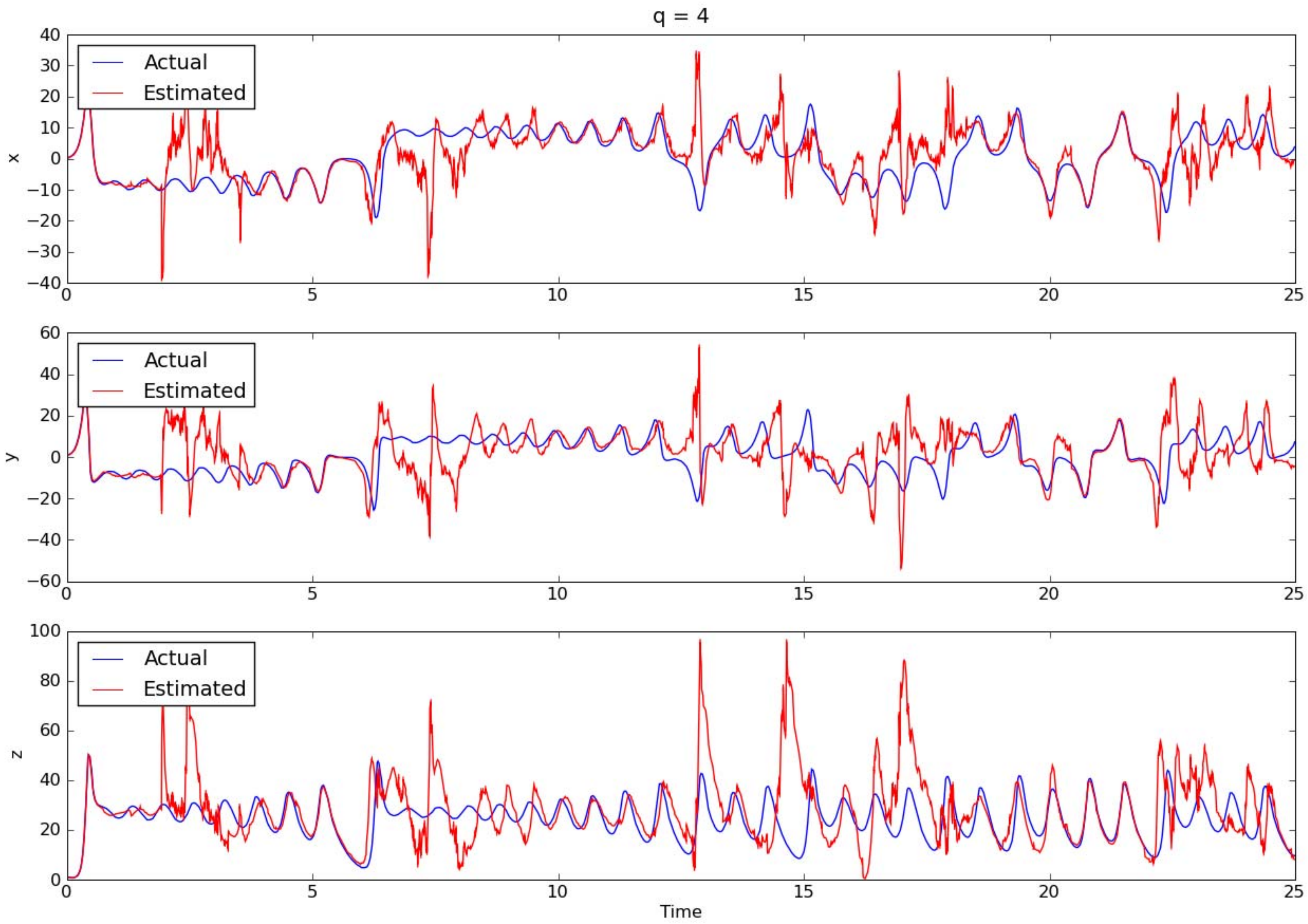
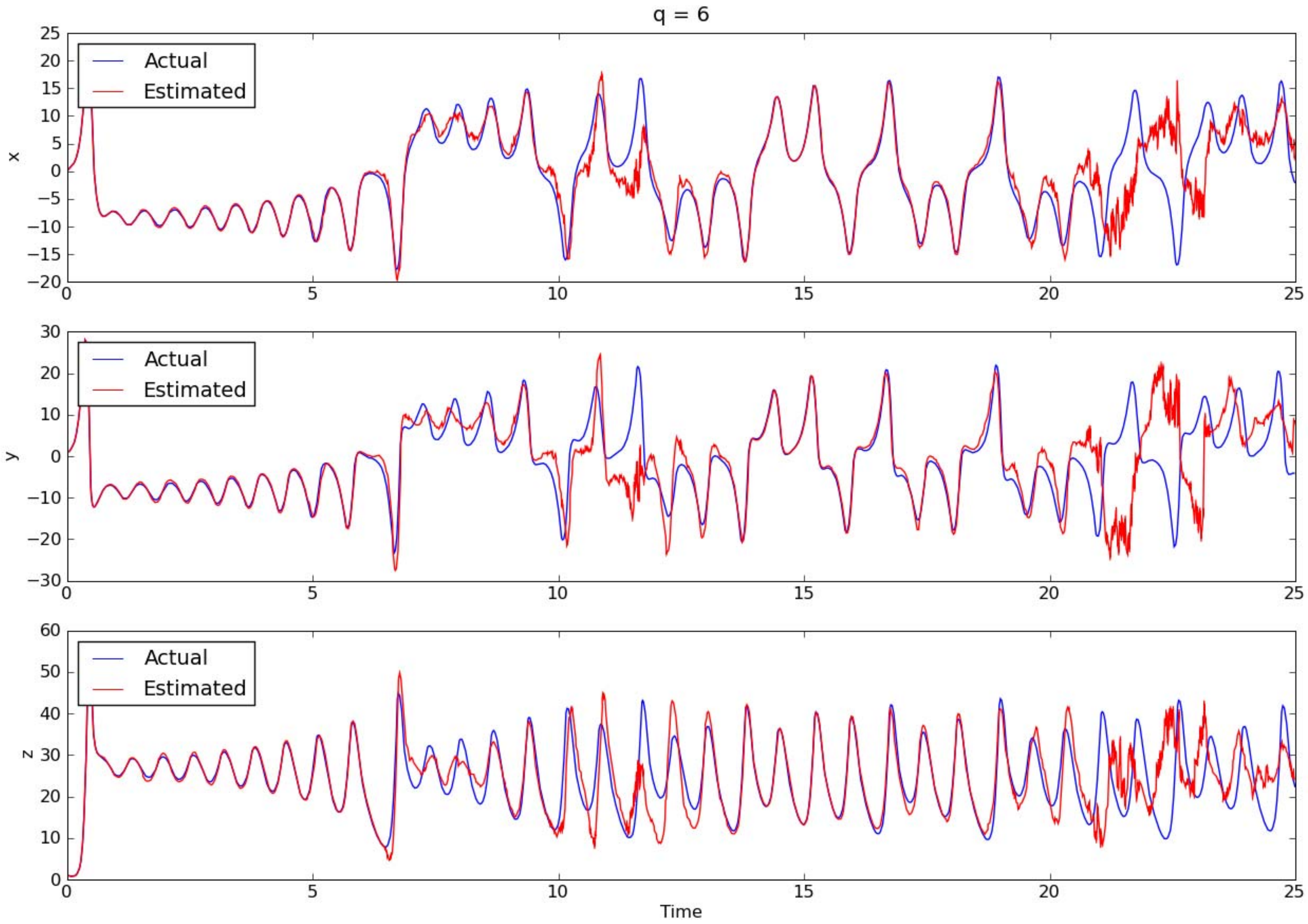
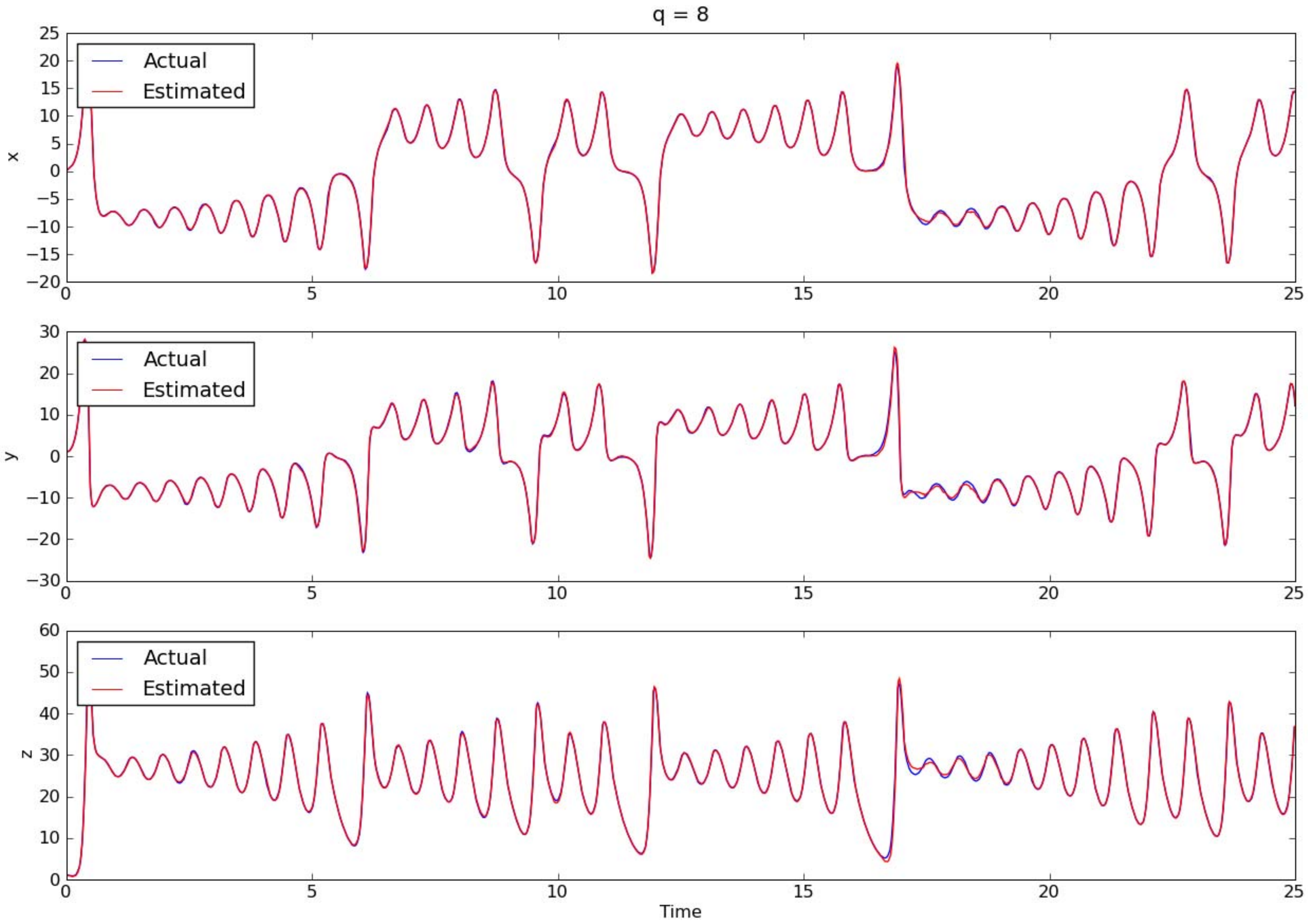
Example – Lorenz system



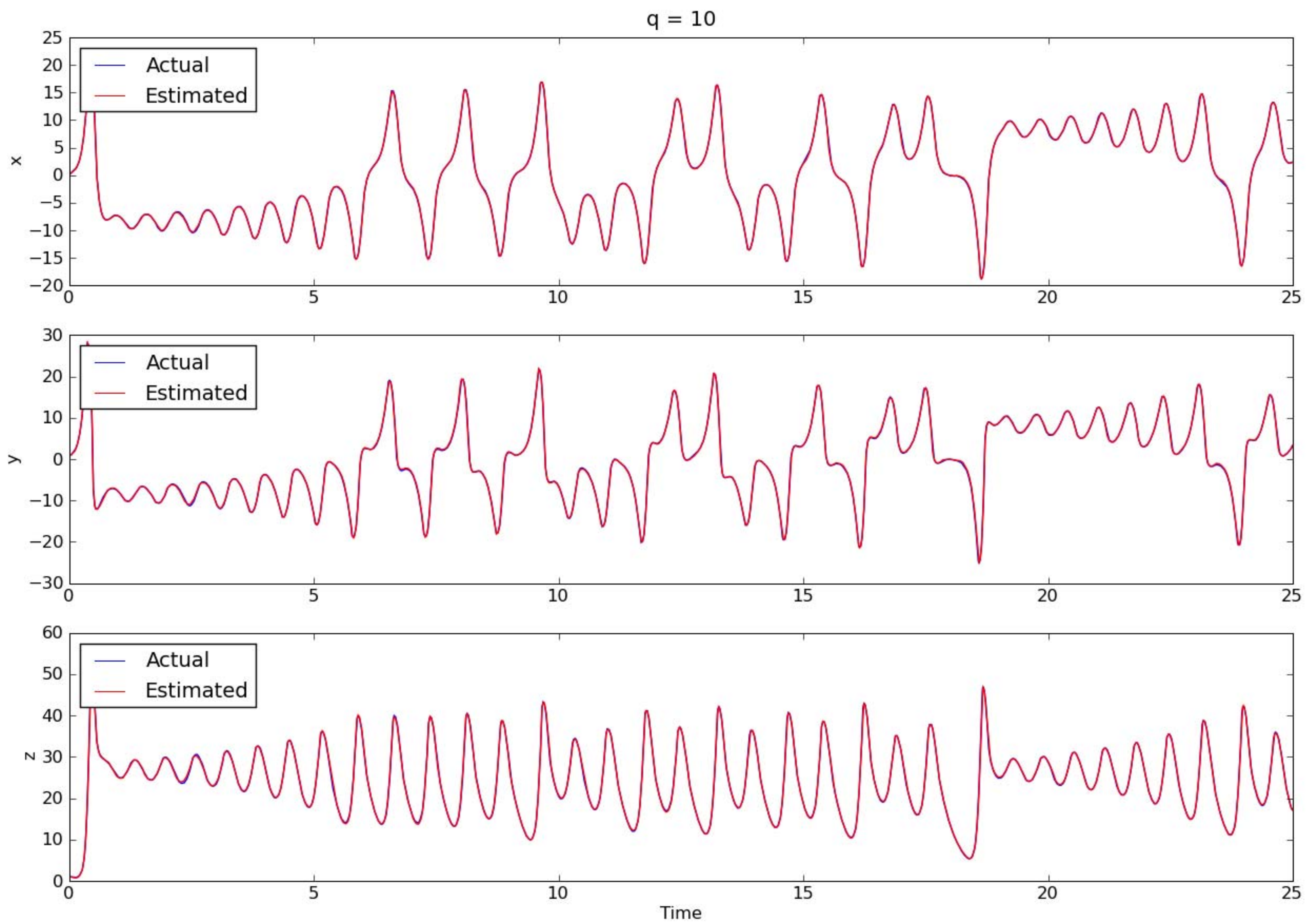
Example – Lorenz system



Example – Lorenz system



Example – Lorenz system



Hardware demonstration

- Sorry! (Tuesday 9:00)

Derivation

- Shamelessly taken from *Applied Optimal Estimation* by Arthur Gelb (1974).

- Goal: Find an unbiased, variance minimizing estimator of the form:

$$\hat{x}_k = L_k' \hat{x}^-{}_k + L_k y_k$$

Derivation - Bias

$$\hat{x}_k = L'_k \hat{x}^-{}_k + L_k y_k$$

$$\hat{x}_k \equiv x_k + \tilde{x}_k$$

$$\hat{x}^-{}_k \equiv x_k + \tilde{x}^-{}_k$$

$$y_k = C_k x_k + v_k$$

$$x_k + \tilde{x}_k = L'_k\left(x_k + \tilde{x}^-{}_k\right) + L_k\left(C_k x_k + v_k\right)$$

$$\tilde{x}_k = \left[L'_k + L_k C_k - I\right]x_k + L'_k \tilde{x}^-{}_k + L_k v_k$$

$$E\left(\tilde{x}_k\right) = 0 \Rightarrow L'_k = I - L_k C_k$$

$$\hat{x}_k = \left(I - L_k C_k\right)\hat{x}^-{}_k + L_k y_k = \hat{x}^-{}_k + L_k\left(y_k - C_k \hat{x}^-{}_k\right)$$

$$\tilde{x}_k = \left(I - L_k C_k\right)\tilde{x}^-{}_k + L_k v_k$$

Derivation – Variance update

$$P_k = E\left(\tilde{x}_k \tilde{x}_k^{\,T}\right) = \left(I - L_k C_k\right)\tilde{x}^-{}_k + L_k v_k$$

$$P_k = E\left\{\left(I - L_k C_k\right)\tilde{x}^-{}_k\left[\tilde{x}^-{}_k{}^T\left(I - L_k C_k\right)^T + v_k{}^T L_k{}^T\right] + L_k v_k\left[\tilde{x}^-{}_k{}^T\left(I - L_k C_k\right)^T + v_k{}^T L_k{}^T\right]\right\}$$

$$E\left(\tilde{x}^-{}_k \tilde{x}^-{}_k{}^T\right) = P_k^-$$

$$E\left(v_k v_k^{\,T}\right) = R_k$$

$$E\left(\tilde{x}^-{}_k v_k^{\,T}\right) = E\left(v_k \tilde{x}^-{}_k{}^T\right) = 0$$

$$P_k = \left(I - L_k C_k\right)P^-{}_k\left(I - L_k C_k\right)^T + L_k R_k L_k{}^T$$

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Derivation – Variance minimization

$$J_k = trace(P_k)$$

$$\frac{\partial J_k}{\partial L_k} = 0$$

$$\frac{\partial}{\partial A}\left(trace\left(ABA^T\right)\right) = 2AB \Rightarrow \frac{\partial J_k}{\partial L_k} = -2\left(I - L_k C_k\right)P_k^{-} C_k^{T} + 2L_k R_k = 0$$

$$L_k = P_k^{-} C_k^{T}\left(C_k P_k^{-} C_k^{T} + R_k\right)^{-1}$$

$$P_k = \left(I - L_k C_k\right)P^{-}{}_k\left(I - L_k C_k\right)^T + L_k R_k L_k^{T}$$

$$P_k = \left(I - L_k C_k\right)P^{-}{}_k$$

Questions?